



1.6.15 The exponential and trigonometric functions

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Introduction

- In this topic, we will
 - Deduce Euler's formula
 - Make some observations on Euler's formula
 - See a geometric interpretation
 - Determine the result when e is raised to a complex power
 - Verify this is likely correct
 - Make some observations
 - Deduce formulas for $\cos(z)$ and $\sin(z)$





Review

- Recall that $|z| \angle \theta = |z| \cos(\theta) + |z| \sin(\theta) j$
 - Thus, $1 \angle \theta = \cos(\theta) + \sin(\theta) j$





Taylor series

- In calculus, you either have seen or will see that the exponential and trigonometric functions can be written as *Taylor series*:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$





Taylor series

- Substitute x with $j\theta$:

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \frac{(j\theta)^7}{7!} + \dots \\ &= 1 + j\theta + j^2 \frac{\theta^2}{2!} + j^3 \frac{\theta^3}{3!} + j^4 \frac{\theta^4}{4!} + j^5 \frac{\theta^5}{5!} + j^6 \frac{\theta^6}{6!} + j^7 \frac{\theta^7}{7!} + \dots \end{aligned}$$

- Recall that, however, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, and so on

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - j \frac{\theta^7}{7!} + \dots$$

- Break this into real and imaginary components.

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right) j$$





Euler's formula

- Let's look at:

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)j$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin(x)$$

- Is it not fair to say $e^{j\theta} = \cos(\theta) + \sin(\theta)j$?
 - Thus, can we not write

$$|z| \angle \theta = |z| \cos(\theta) + |z| \sin(\theta)j = |z| e^{j\theta}$$





Observation

Theorem: $|e^{j\theta}| = 1$

Proof: As $e^{j\theta} = \cos(\theta) + \sin(\theta)j$, it follows that

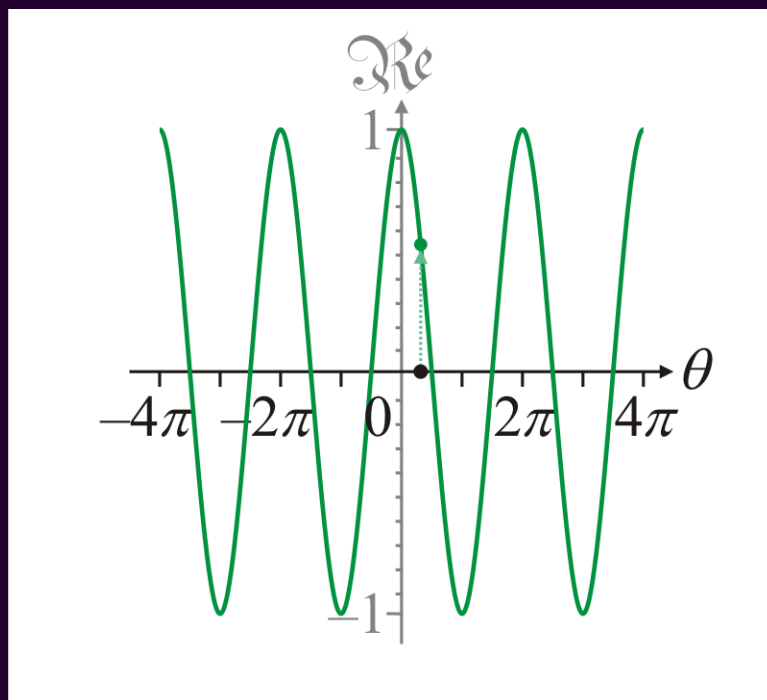
$$\begin{aligned}|e^{j\theta}| &= |\cos(\theta) + \sin(\theta)j| \\ &= \sqrt{\cos(\theta)^2 + \sin(\theta)^2} \\ &= \sqrt{1} = 1 \quad \text{QED}\end{aligned}$$





Geometric interpretation

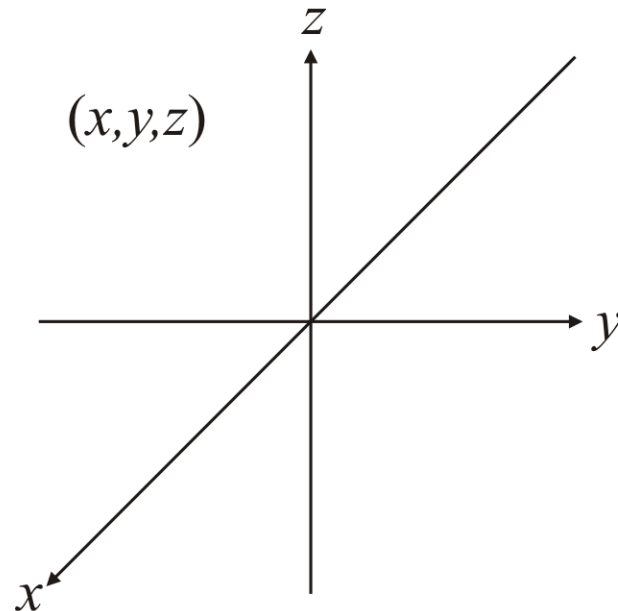
- Plotting $\cos(\theta)$ has you plot the value of $\cos(\theta)$ for all values on the abscissa
 - In this case, $-4\pi \leq \theta \leq 4\pi$
 - You can read off the value, for example, $\cos(1)$





Geometric interpretation

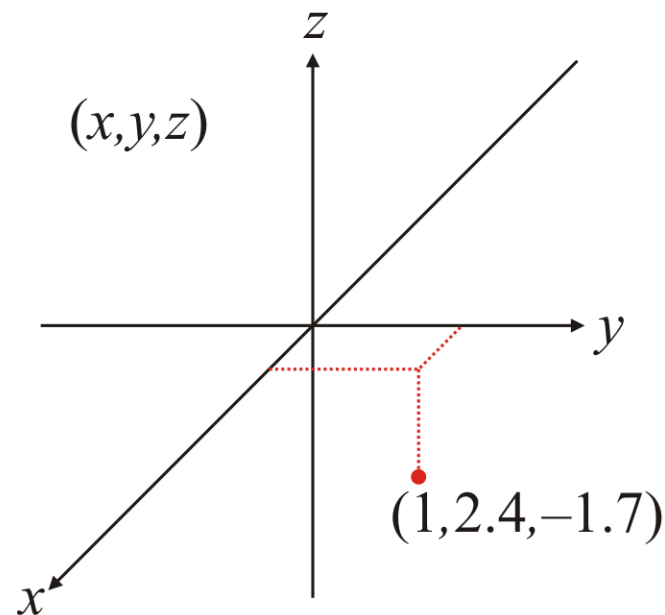
- Now, in plotting in three dimensions, we must project onto a plane
 - We use the right-hand rule to determine the axes





Geometric interpretation

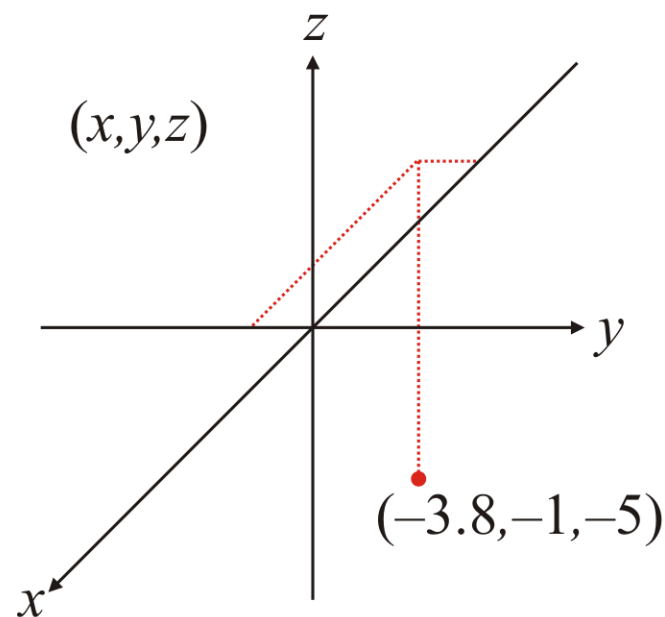
- To plot a point in three dimensions, you need a hint as to where that point is relative to the axes





Geometric interpretation

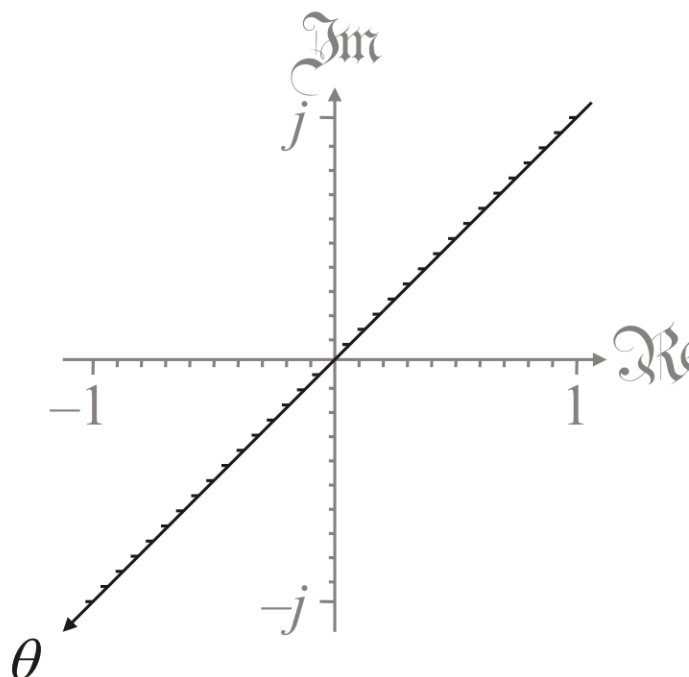
- This same point could represent many different values!
 - This is the result of us projecting 3-space onto a plane





Geometric interpretation

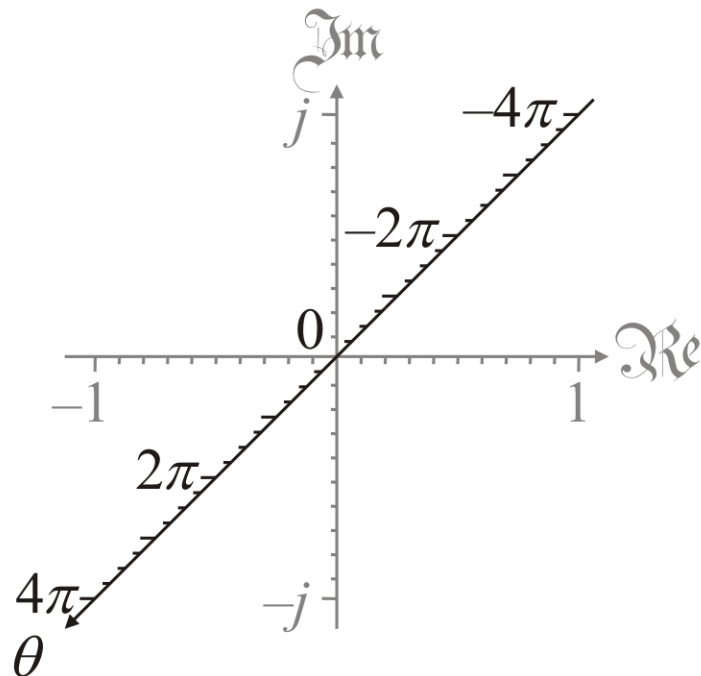
- When plotting a complex-valued function f of a real variable, the three axes will represent:
 - The independent variable θ
 - The real component of $f(\theta)$
 - The imaginary component of $f(\theta)$





Geometric interpretation

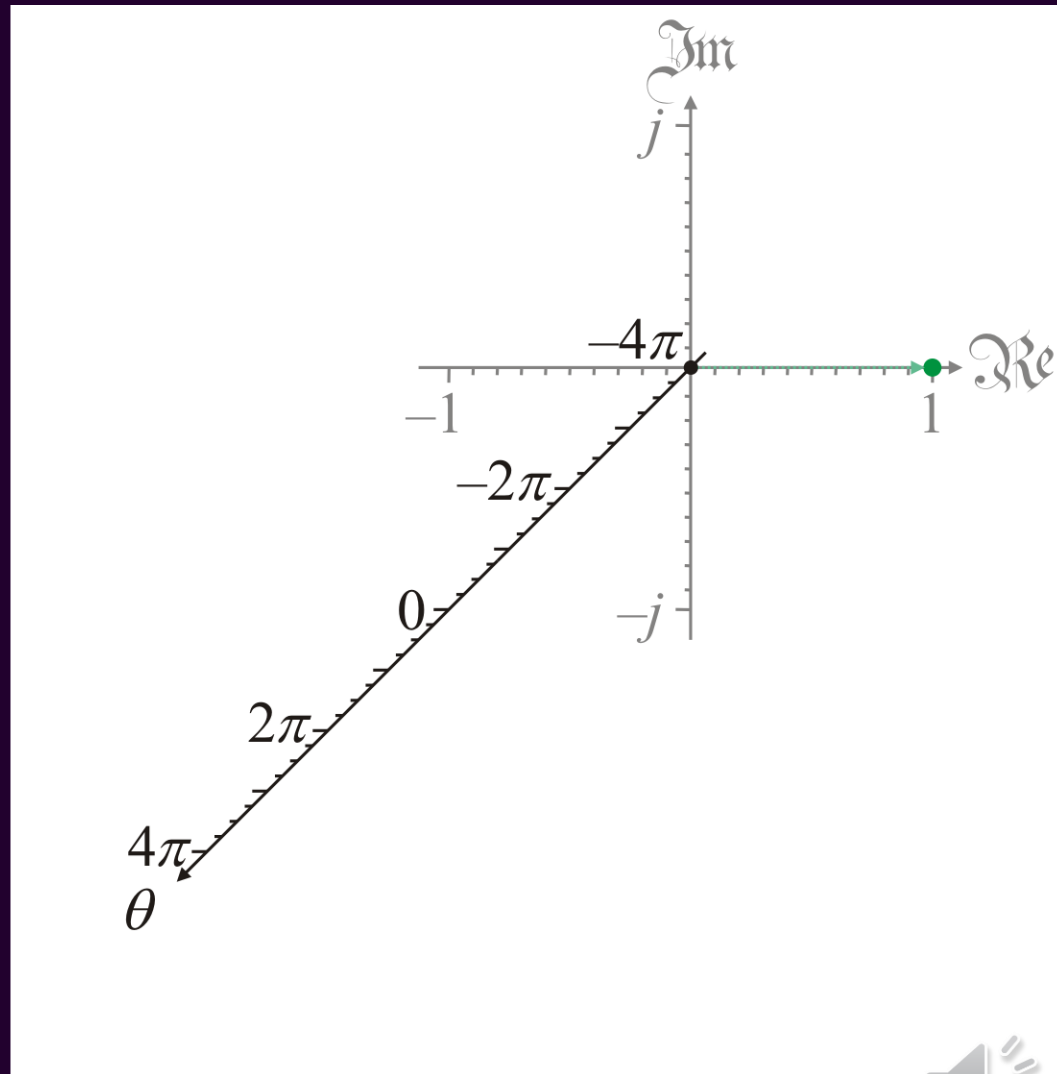
- Suppose we want to plot a function f on $-4\pi \leq \theta \leq 4\pi$
 - We will try this first for $\cos(\theta)$ and then $e^{\theta j}$





Geometric interpretation

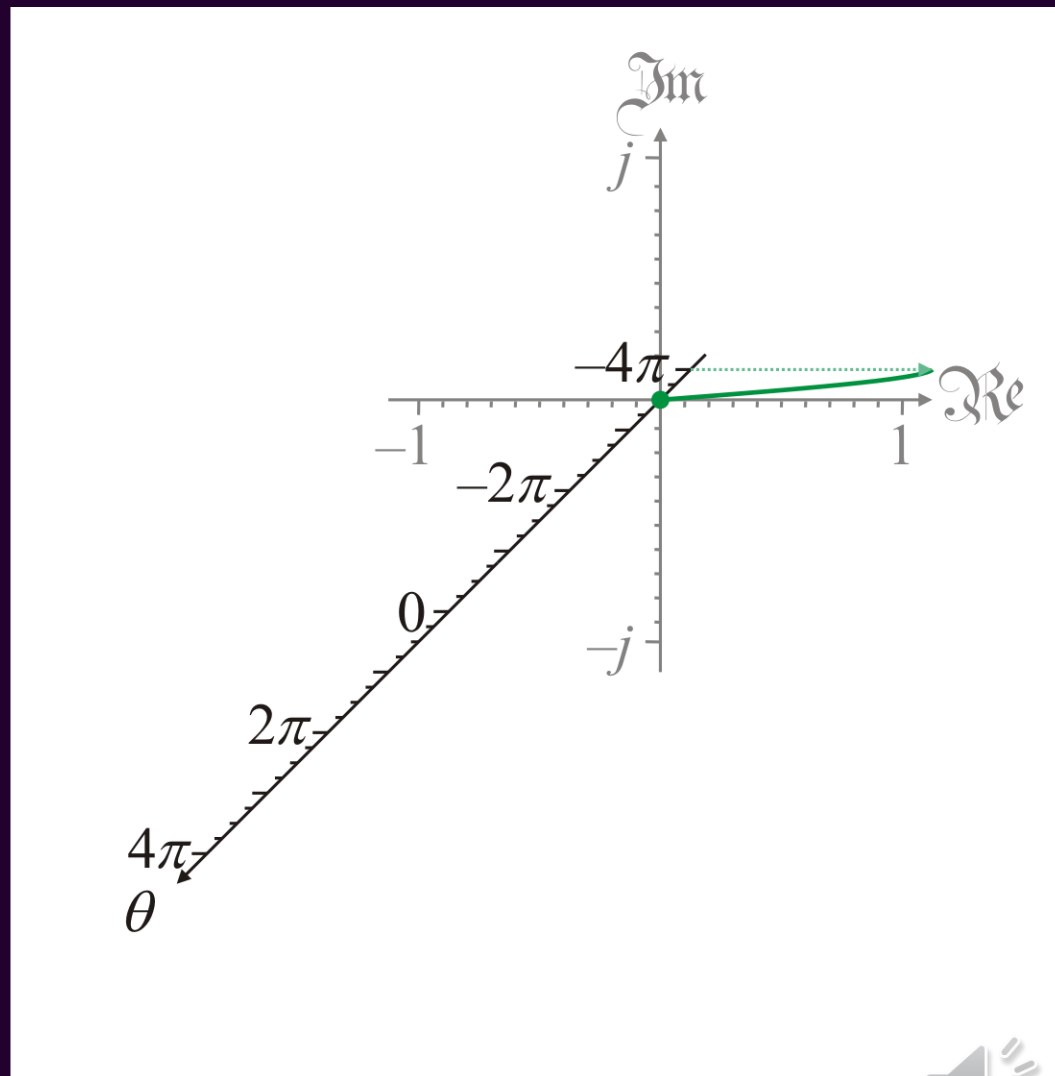
- First, we note that a function $\cos(-4\pi) = 1$
 - We emphasize this with an arrow





Geometric interpretation

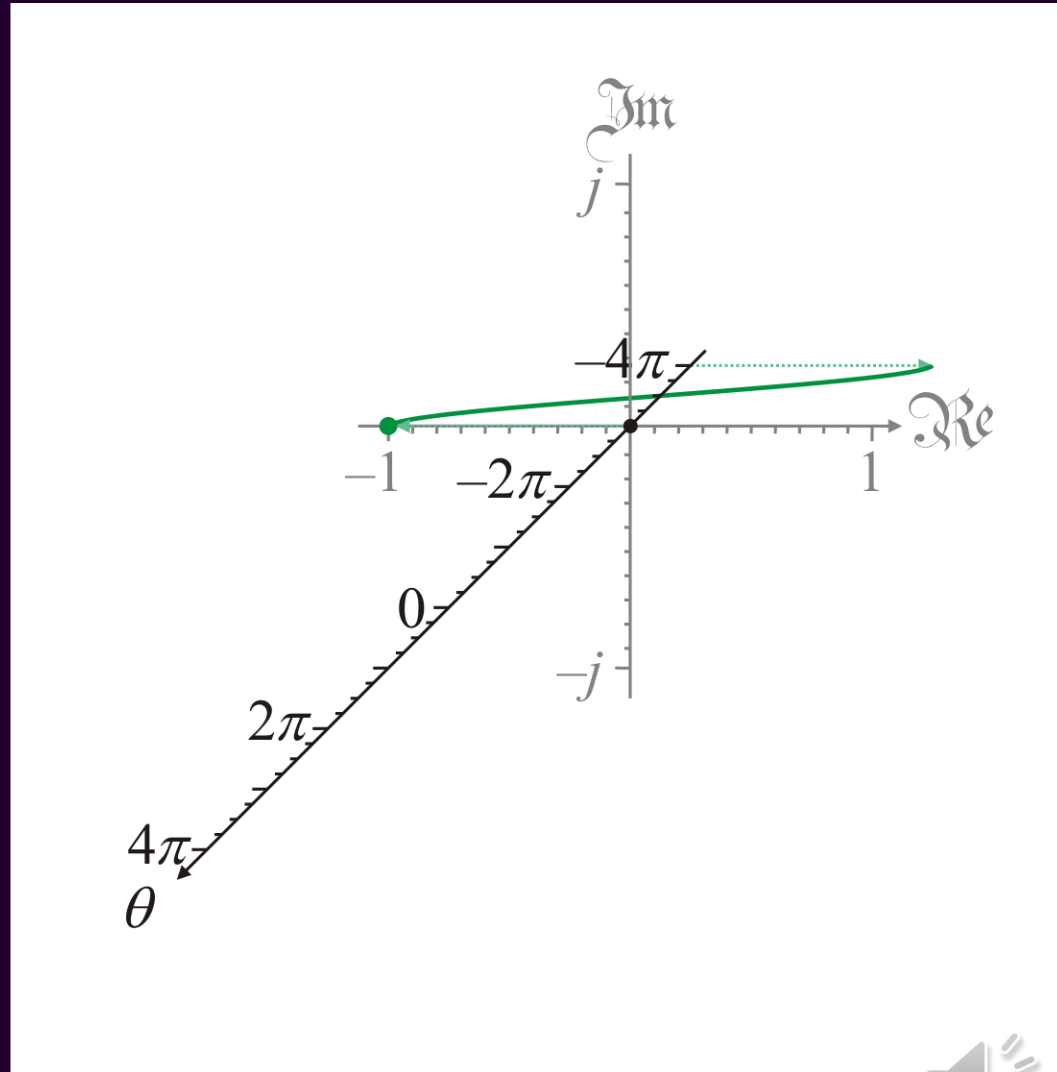
- Next, we note that $\cos(-3.5\pi) = 0$





Geometric interpretation

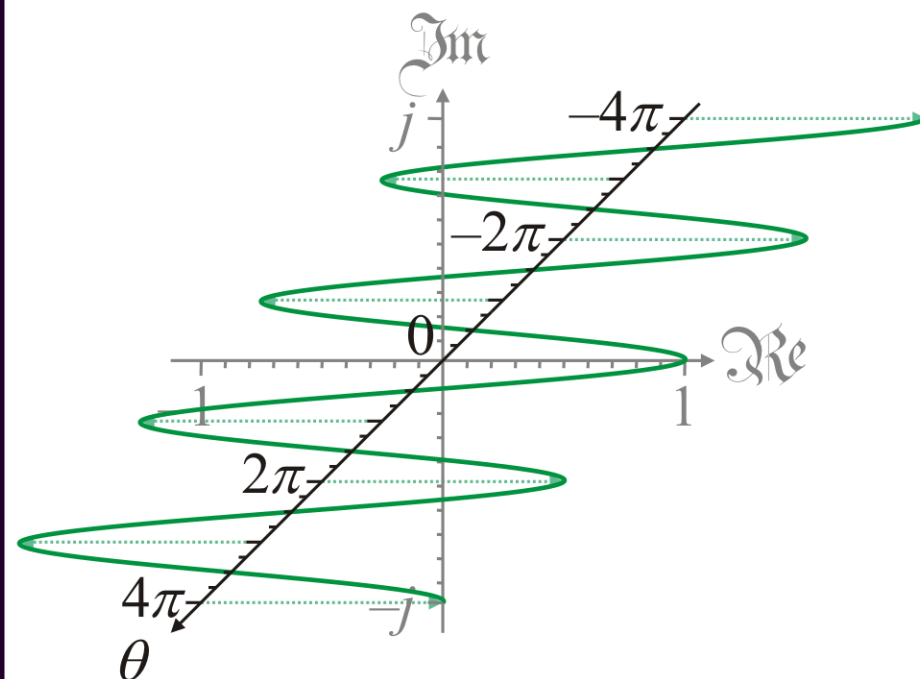
- Next, we note that $\cos(-3\pi) = -1$





Geometric interpretation

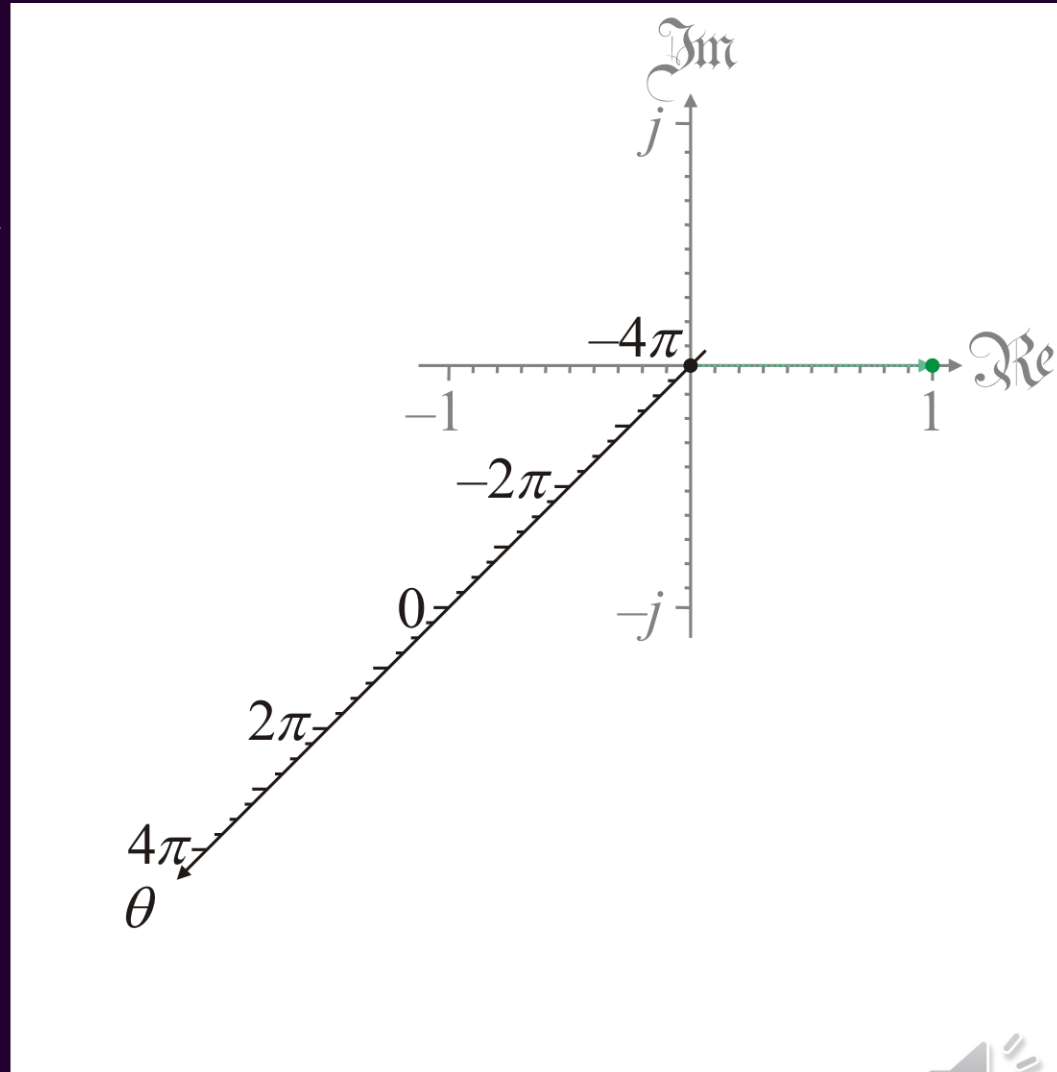
- Continuing this, we can plot the cosine function for all values in question
 - The imaginary part is always zero
 - All points lie in the xy -plane





Geometric interpretation

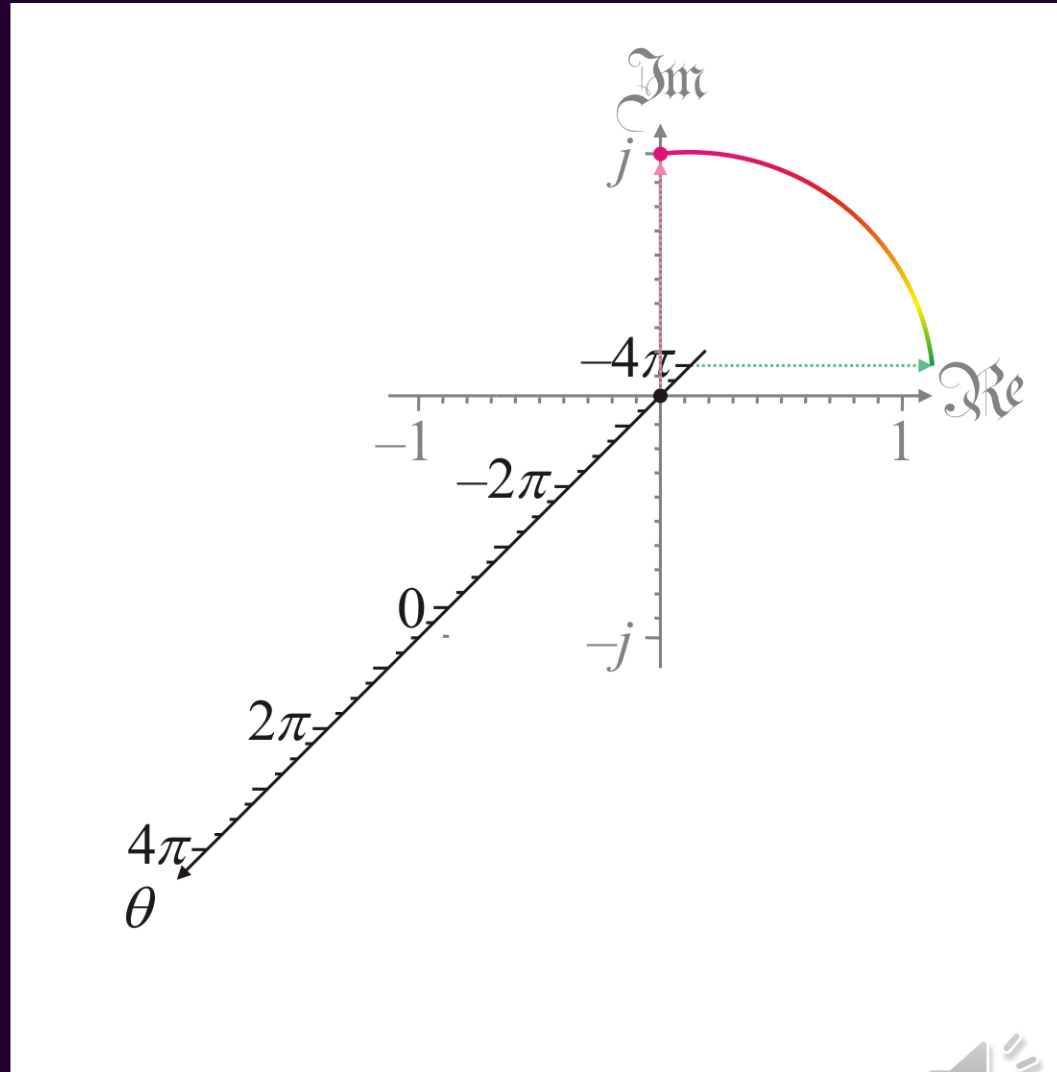
- Next, let's plot $e^{\theta j}$
 - This is a complex-valued function of a real variable
 - We note $e^{-4\pi j} = 1$





Geometric interpretation

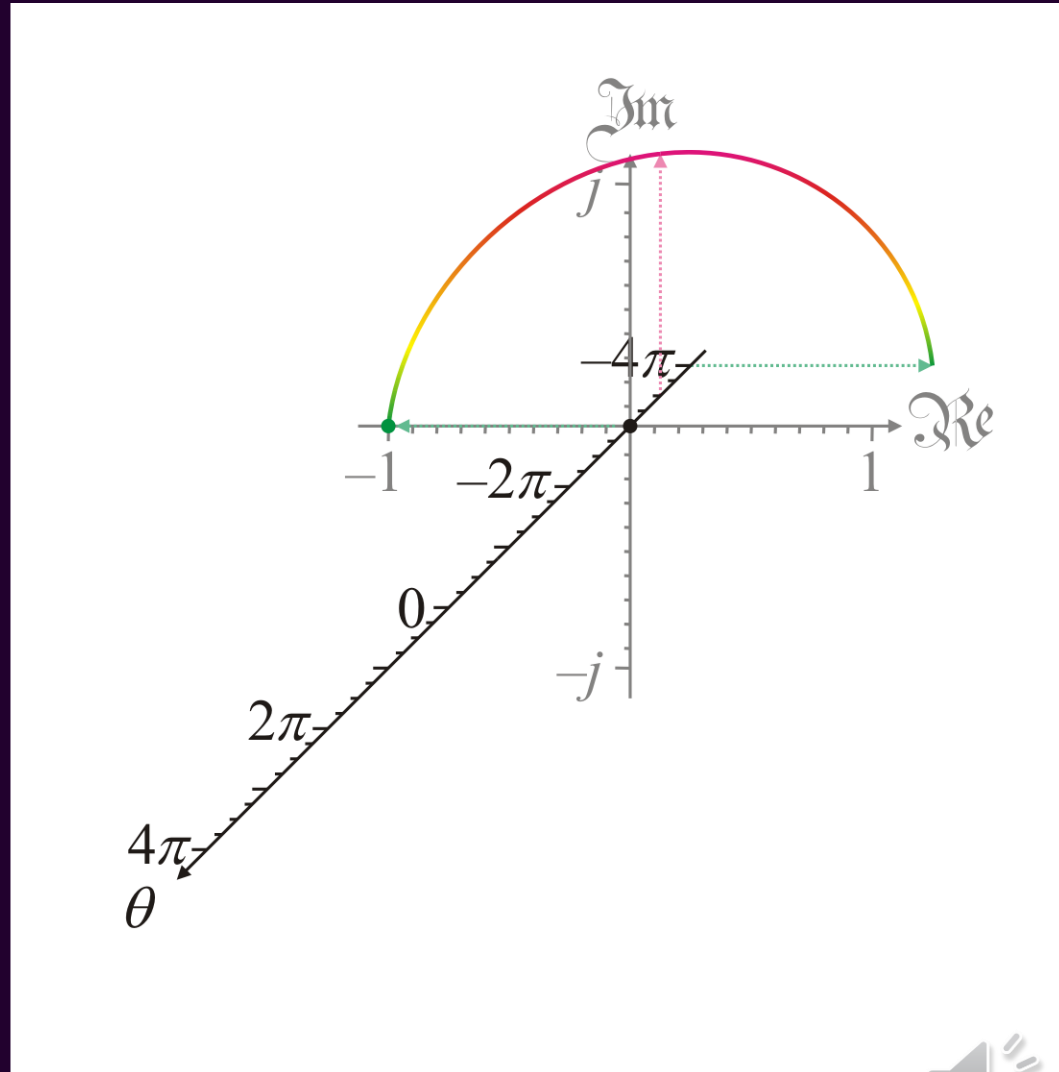
- Next, $e^{-3.5\pi j} = j$
 - We use color to highlight the value of the imaginary component:
 - Green is close to zero
 - Magenta is the most positive
 - Purple is the most negative





Geometric interpretation

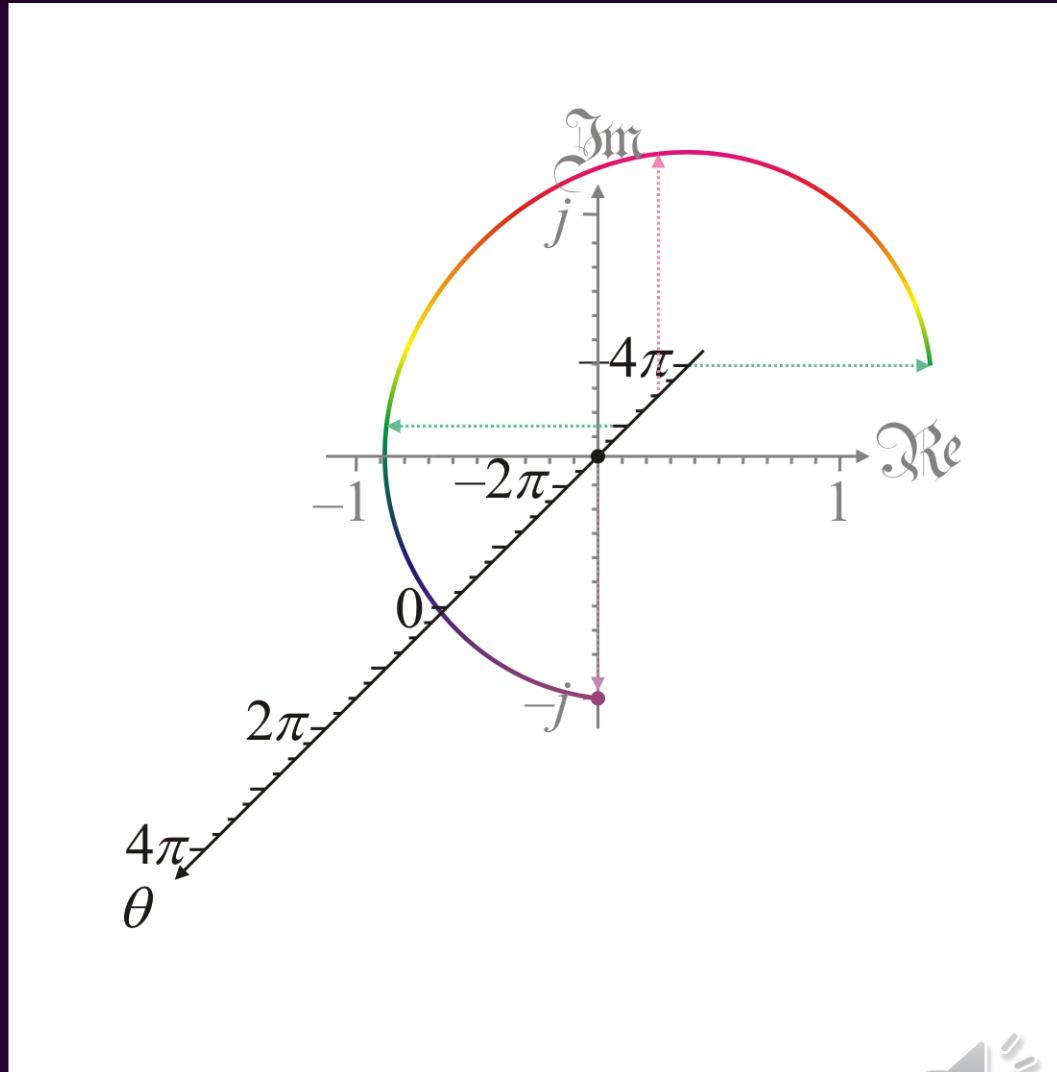
- Next, $e^{-3\pi j} = -1$





Geometric interpretation

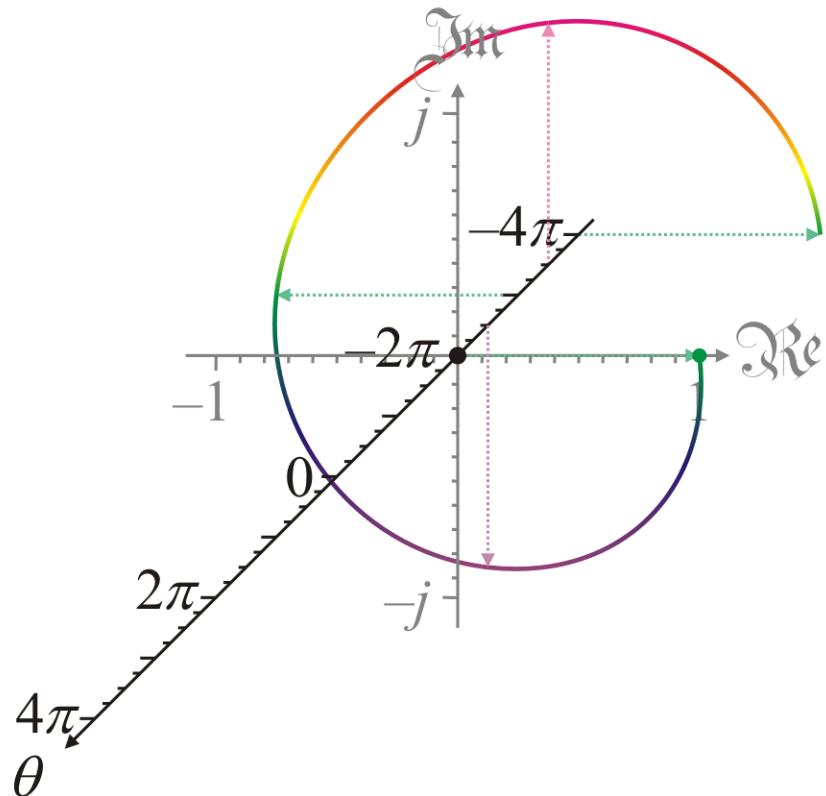
- Next, $e^{-2.5\pi j} = -j$





Geometric interpretation

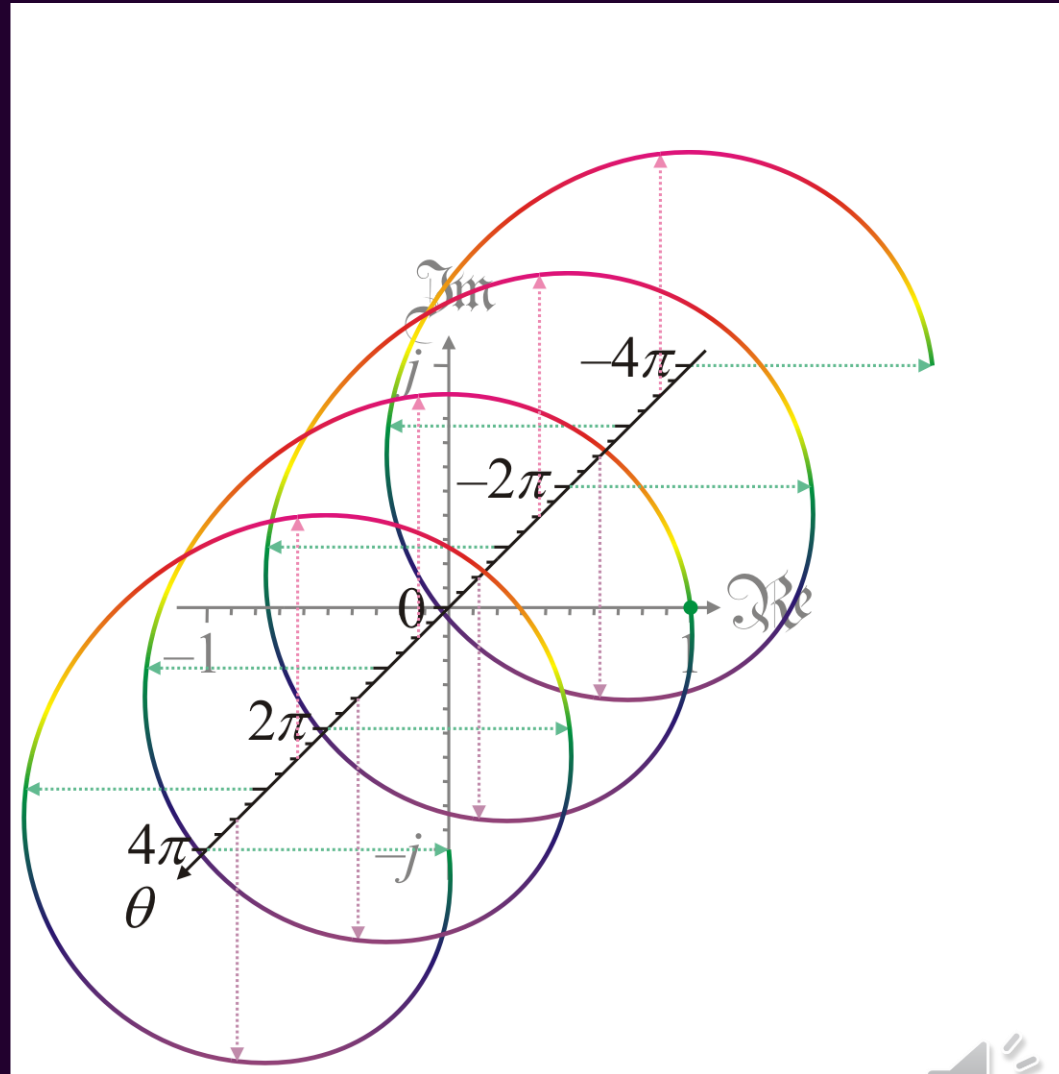
- Next, $e^{-2\pi j} = 1$
 - Repeating this, we note
 - $e^{-1.5\pi j} = j$
 - $e^{-\pi j} = -1$
 - $e^{-0.5\pi j} = -j$
 - $e^{-0\pi j} = 1$





Geometric interpretation

- Plotting $e^{\theta j}$ for more values of θ reveals a corkscrew motion
 - Recall that
$$\lim_{x \rightarrow \infty} e^x = \infty$$
$$\lim_{x \rightarrow -\infty} e^x = 0$$
 - Here, we see that $e^{\theta j}$ corkscrews through the complex plane for all θ





Complex exponential

- Question?
 - If $z = \alpha + \beta j$, what is e^z ?
- Solution:
 - From secondary school, you know that
$$e^{x+y} = e^x e^y$$
 - Thus, $e^{\alpha+\beta j} = e^\alpha e^{\beta j}$
$$= e^\alpha (\cos(\beta) + \sin(\beta) j)$$





Complex exponential

- As an example, let us try this out:
 - For $z = 0.09 - 0.3j$, calculate and approximate e^z using the formula and the Taylor series:

$$\begin{aligned}e^{0.09-0.3j} &= e^{0.09} (\cos(-0.3) + \sin(-0.3)j) \\&\approx 1.0941742837052(0.9553364891256 - 0.2955202066613j) \\&\approx 1.0453046186865 - 0.3233506104441j\end{aligned}$$





Complex exponential

- As an example, let us try this out:
 - Using the 5th-order Taylor series approximation, we get that

$$\begin{aligned} e^{0.09-0.3j} &\approx 1 + (0.09 - 0.3j) + \frac{(0.09 - 0.3j)^2}{2!} + \frac{(0.09 - 0.3j)^3}{3!} + \frac{(0.09 - 0.3j)^4}{4!} + \frac{(0.09 - 0.3j)^5}{5!} \\ &= 1.09 - 0.3j + \frac{-0.819 - 0.54j}{2!} + \frac{-0.023571 + 0.01971j}{3!} + \frac{0.00379161 + 0.0088452j}{4!} \\ &\quad + \frac{0.0000249567075 - 0.2845125j}{5!} \\ &= 1.09 - 0.3j + (-0.04095 - 0.027j) + (-0.0039285 + 0.003285j) \\ &\quad + (0.00015798375 + 0.00036855j) + (0.0000249567075 + 0.000002845125j) \\ e^{0.09-0.3j} &\approx 1.0453044404575 - 0.323349295125j \end{aligned}$$





Complex exponential

- Let us compare our two results:
 - Formula:

$$e^{0.09-0.3j} \approx 1.0453046186865 - 0.3233506104441j$$

- Taylor series approximation

$$e^{0.09-0.3j} \approx 1.0453044404575 - 0.3233492951250j$$





Observation

Theorem: $(e^z)^* = e^{z^*}$

Proof: If $z = \alpha + \beta j$, then

$$\begin{aligned}(e^z)^* &= (e^{\alpha + \beta j})^* \\&= (e^\alpha e^{\beta j})^* \\&= (e^\alpha)^* (e^{\beta j})^* \\&= e^\alpha (\cos(\beta) + \sin(\beta) j)^* \\&= e^\alpha (\cos(\beta) - \sin(\beta) j) \\&= e^\alpha (\cos(-\beta) + \sin(-\beta) j) \\&= e^\alpha e^{-\beta j} = e^{\alpha - \beta j} = e^{z^*} \quad \text{QED}\end{aligned}$$





Observation

Theorem: $(e^{\theta j})^* = e^{-\theta j}$

Proof: A consequence of the previous theorem.





Euler's identity

Theorem: $e^{\pi j} + 1 = 0$

Proof:
$$\begin{aligned} e^{\pi j} + 1 &= (\cos(\pi) + \sin(\pi)j) + 1 \\ &= (-1 + 0 \cdot j) + 1 \\ &= -1 + 1 = 0 \quad \mathcal{QED} \end{aligned}$$





Trigonometric functions

- What are $\cos(z)$ and $\sin(z)$?
 - We know for a real θ , $e^{\theta j} = \cos(\theta) + \sin(\theta)j$
 - Posit that $e^{zj} = \cos(z) + \sin(z)j$ continues to be true if z is a complex number
 - Thus, $e^{zj} = \cos(z) + \sin(z)j$
 $e^{-zj} = \cos(z) - \sin(z)j$

 $e^{zj} + e^{-zj} = 2\cos(z)$
 - Thus, $\cos(z) = \frac{e^{zj} + e^{-zj}}{2}$
 - Similarly, $\sin(z) = \frac{e^{zj} - e^{-zj}}{2j}$





Summary

- In this topic, we described how to calculate the exponential function when the argument is complex
 - We deduce Euler's formula $e^{\theta j} = \cos(\theta) + \sin(\theta)j$
 - We saw that $|z| \angle \theta = |z|e^{\theta j}$
 - We determined formula for e^z
 - We saw how to plot a complex-valued function of a real variable
 - $e^{\theta j}$ produces a spiral in the complex plane
 - We checked the formulas are likely correct
 - Made some observations, including Euler's identity





References

- [1] https://en.wikipedia.org/wiki/Euler%27s_formula
- [2] https://en.wikipedia.org/wiki/Euler%27s_identity





Acknowledgments

Discussions with and courses from Prof. John A.R. Holbrook.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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